Estimation metrics and optimal regularization in a Tikhonov digital filter for the inverse heat conduction problem

Keith A. Woodbury[*a], James V. Beck*[b]*

[a] Mechanical Engineering Department, The University of Alabama, Tuscaloosa, AL, 35487, USA
[b] Department of Mechanical Engineering, Michigan State University, East Lansing, MI 48824, USA

1 Abstract

Tikhonov regularization for the inverse heat conduction problem (IHCP) is considered a "whole domain" or "batch" method, meaning that observations are needed over the entire time domain of interest, and that calculations must be performed all-at-once in a batch. This paper examines the structure of the Tikhonov regularization problem and concludes that the method can be interpreted as a sequential filter formulation for continuous processing of data. Several general observations regarding features of the filter formulation are noted. Two error norms are discussed: one regarding temperature and one regarding heat flux. It is shown that these metrics can be split into two parts: one dependent on the heat flux history (bias error) and one dependent on the measurement noise (random error). Two examples demonstrate that the optimal selection of the regularization parameter to minimize the heat flux error yields results similar to the classical Morozov principle defined through temperature error, and that the results are relatively insensitive to the precise selection of the parameter.

2 Structure of the Tikhonov Filter

The structure of the filter matrix **F** is not readily apparent. To see its characteristics, a specific instance of this matrix is calculated for the X22B10T0 case for $N = 121$ time steps, dimensionless $\Delta t = 0.05$, $x_1 = 1.0$, and α_0=0.01 using piecewise constant interpolation to represent the heat flux. Several rows of the resulting **F** matrix are plotted in Fig. 1. The values for row 40 are plotted with a dashed heavy line to illustrate the shape of the curve for one row. Except for the first few (about five or ten in this example) and last several (about ten or so for this case), all the rows are seen to have the same entries but are shifted in time. Figure 1 gives insight to the known fact that the TR method gives poor results at the beginning and, especially, at the end of the time window. The degeneracy of the filter coefficients near the end of the time explains this phenomenon. It is also apparent from Fig. 1 that the filter coefficients drop to zero for some relatively small time window surrounding any current time, t_M. These observations lead to an idealization of the **F** matrix, ignoring the anomalies on the first few and last several rows. The matrix exhibits a banded structure which is symmetric about the diagonal and is Toeplitz in structure. Post multiplication of a Toeplitz matrix (here **F**) any vector (say, **g**) has a discrete convolution form:

$$\left[\mathbf{Fg}\right]_{\text{row } i} = \sum_{j=0}^{N-1} f_{i-j} g_j = \left[\mathbf{F} * \mathbf{g}\right]_i \tag{2}$$

3 Heat Flux error estimates

The goal of the IHCP is to determine, as accurately as possible, the surface heating action at the boundary. Hence, it is reasonable to consider the error in the heat flux as a metric to measure the success

---

[*] Corresponding author. E-mail addresses: keith.woodbury@ua.edu; beck@msu.edu

of the estimation. However, heat flux error estimates are impractical to evaluate in experimental settings, because the exact value of the heat flux, **q**, is unknown, but these error estimates are useful in a numerical setting and, as will be seen, offer a mechanism to optimize an IHCP regularization method and to compare different IHCP techniques.

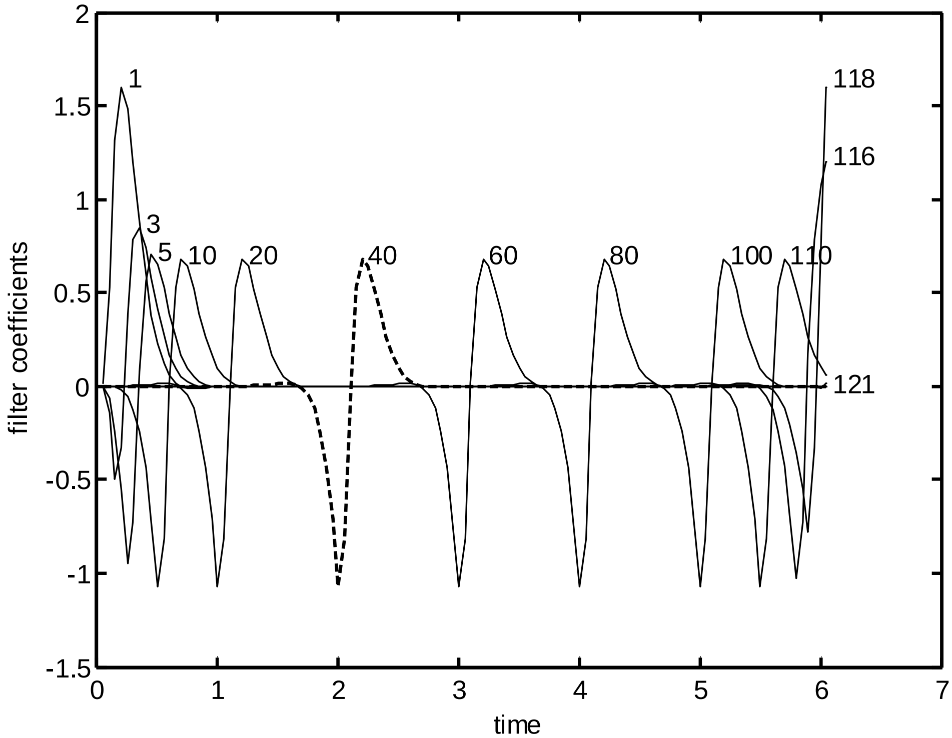

Figure 1. Rows of the **F** matrix plotted as functions of time. $\alpha_0 = 0.01$, $x = 1.0$, $N = 121$, $\Delta t = 0.05$. (tikhonov_filter.m)

The residual sum of squares in the errors in the heat flux is

$$R_q = (\hat{\mathbf{q}} - \mathbf{q})^T (\hat{\mathbf{q}} - \mathbf{q}) = (\mathbf{FY} - \mathbf{q})^T (\mathbf{FY} - \mathbf{q}) \tag{36}$$

This residual sum of squares can be computed using Monte Carlo methods. A more efficient approach is to minimize its expected value. The expected value of the residual sum of squares in the errors in the heat flux is

$$E(R_q) = E[(\mathbf{FY})^T \mathbf{FY}] - [\mathbf{F}E(\mathbf{Y})]^T \mathbf{q} - \mathbf{q}^T \mathbf{F}E(\mathbf{Y}) + \mathbf{q}^T \mathbf{q}] \tag{37}$$

Employing the standard statistical assumptions, and following a derivation similar to that for $E(R_T)$ this expected value has two parts. The important result is

$$E(R_q) = [(\mathbf{FX} - \mathbf{I})\mathbf{q}]^T [(\mathbf{FX} - \mathbf{I})\mathbf{q}] + \sigma_Y^2 tr(\mathbf{F}^T \mathbf{F}) = E_{q,bias} + E_{q,rand} \tag{38}$$